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Conformally Flat Charged Dust

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It is proved that the most general conformally flat singularity-free solution for charged dust in static equilibrium must be spherically symmetric and the solution is unique. It matches with exterior Reissner–Nordström metric. The manifest form of the metric conformal to Minkowskian metric is also given.

1. INTRODUCTION

There is at present a good amount of literature in general theory of relativity on static charged dust in equilibrium. There are a few general theorems as well as some special solutions for such a distribution. One of the interesting results about charged dust in static equilibrium obtained by De and Raychaudhuri (1968) is that the density of charge $|\sigma|$ is equal to the density of mass ρ in relativistic units. The metric then belongs to the interior Papapetrou–Majumdar class (Papapetrou, 1947; Majumdar, 1947; Das, 1962). One can generate special classes of solutions in this case choosing either a physically reasonable matter density or the metric satisfying suitable boundary conditions. Explicit solutions for charged dust in equilibrium ($\rho=|\sigma|$) have been previously discussed in spherical and also in spheroidal symmetries by some workers (Bonnor, 1965; Bonnor and Wickramsuriya, 1975; Cooperstock and De La Cruz, 1978). We have in this paper proved an interesting theorem about static charged dust that the most general conformally flat charged dust in static equilibrium must be spherically symmetric and the solution is unique. It can also be matched with the outside electrovac.

In Section 2 there are some general considerations, where we have obtained the exact form of g_{00} for the conformally flat metric having

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spatial part completely Euclidean. The solution is obtained from the vanishing of the Weyl tensor alone. In Section 3 we have used this result to get the most general conformally flat solution of the Einstein–Maxwell equations corresponding to static charged dust in equilibrium irrespective of its symmetry.

2. GENERAL CONSIDERATIONS

We start with a metric in the form

$$ds^2 = e^{2\nu}\,dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

and assume that ν is a function of all the space and time coordinates. It is interesting to find conditions on the function ν in order that the metric (1) is conformally flat. In fact, the conformal flatness demands a specific form for ν involving in general five arbitrary functions of time. The only nonvanishing Riemann–Christoffel curvature tensor component is

$$R^0_{\kappa l 0} = \nu_{\kappa l} + \nu_\kappa \cdot \nu_l \tag{2}$$

where Latin indices stand for 1, 2, 3, and $x^0 = t$. The subscripts indicate ordinary differentiation with respect to the corresponding coordinates. The nonvanishing Ricci tensors and the Ricci scalar are

$$R_{\kappa l} = \nu_{\kappa l} + \nu_\kappa \nu_l \tag{3}$$

$$R_{00} = -e^{2\nu} \sum_{\kappa=1}^{3} \left(\nu_{\kappa\kappa} + \nu_\kappa^{\,2}\right) \tag{4}$$

and

$$R = -2 \sum_{\kappa=1}^{3} \left(\nu_{\kappa\kappa} + \nu_\kappa^{\,2}\right) \tag{5}$$

The general expression for the Weyl tensor (Eisenhart, 1926) is

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \left\{g_{\mu[\beta}R_{\alpha]\nu} + g_{\nu[\alpha}R_{\beta]\mu}\right\} - \tfrac{1}{3}R g_{\mu[\alpha}g_{\beta]\nu} \tag{6}$$

In the above we have defined the Ricci tensor $R_{\mu\nu}$ as

$$R_{\mu\nu} = g^{\alpha\beta}R_{\beta\mu\nu\alpha}$$

Now in view of (2), (3), (4), and (5) the vanishing of the component $C_{0\kappa l0}$ of the Weyl tensor leads us to

$$(e^{\nu})_{,\kappa l}=\tfrac{1}{3}\delta_{\kappa l}\nabla(e^{\nu}) \tag{7}$$

The symbol ∇ stands for the three-dimensional ordinary Laplacian. Vanishing of all other components of the Weyl tensor leads us either to identity relations or to the same relation (7). In fact, from (7) one can easily conclude by elementary arguments that

$$(e^{\nu})_{,\kappa l}=0, \qquad \kappa\neq l \tag{8a}$$

and

$$(e^{\nu})_{,11}=(e^{\nu})_{,22}=(e^{\nu})_{,33}=f(t) \tag{8b}$$

$f(t)$ being an arbitrary function of time. The solution for e^{ν} can be immediately arrived at and may be expressed as

$$e^{\nu}=\tfrac{1}{2}f(t)(x^2+y^2+z^2)+a(t)x+b(t)y+c(t)Z+g(t) \tag{9}$$

where a, b, c, and g are other arbitrary functions of time.

3. CONFORMALLY FLAT STATIC CHARGED DUST

In view of the results obtained in Section 2 we arrive in this section at the most general class of solutions for conformally flat static charged dust. There are many interesting results in the literature about charged dust in static equilibrium. These belong to the interior Papapetrou–Majumdar class mentioned earlier. All such metrics can be expressed in the form

$$ds^2=e^{\nu}\,dt^2-e^{-\nu}(dx^2+dy^2+dz^2) \tag{10}$$

and the whole set of field equations in this case reduces to

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-4\pi\rho(1+V)^3 \tag{11}$$

where $e^{\nu}=(1+V)^{-2}$ and ρ is the mass density of the dust. ρ vanishes in the outside empty space. The function ν, however, is a function of space coordinates only for static configuration. One may at this stage attempt to find the most general conformally flat exact solutions of Einstein–Maxwell equations corresponding to charged dust in equilibrium without any restriction on the symmetry of the dust distribution. To do this one must restrict the function ν in (10) so that the Weyl tensor vanishes. The line element (10) now can be conveniently written in the form

$$ds^2 = e^{-\nu}\left[e^{2\nu}dt^2 - (dx^2 + dy^2 + dz^2)\right] \tag{12}$$

From the well-known properties of conformally flat metrics it follows that the metric represented by the expression within the parentheses in (12) is also conformally flat and one immediately gets the solution for e^ν in the expression (9) with f, a, b, c, and g, however, being constants, because of the static character of the metric. It can further be shown by elementary considerations that coordinate transformations like

$$x' = x + x_0, \qquad y' = y + y_0, \qquad z' = z + z_0$$

reduce the metric to be a function of the radial coordinate alone. In the above x_0, y_0, and z_0 are, however, constants. The solution finally appears in the form

$$e^\nu = (Ar^2 + B) \tag{13}$$

where A and B are constants. The line element (12) can now be written in the form

$$ds^2 = (Ar^2 + B)\,dt^2 - (Ar^2 + B)^{-1}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) \tag{14}$$

One now arrives at a quite general results that conformally flat static charged dust distribution in equilibrium can only be spherically symmetric and further the only such solution is given by (14). Now the metric in the exterior region in this case is the special case of the Reissner–Nordström metric, which is given by

$$ds^2 = (1 + m/r)^{-2}\,dt^2 - (1 + m/r)^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) \tag{15}$$

Here m is the mass or charge parameter, both of them being equal in this case. The interior metric (14) can be matched with the exterior metric (15) at the boundary $r = r_0$ provided

$$A = \frac{m/r_0{}^3}{(1 + m/r_0)^3} \quad \text{and} \quad B = \frac{1}{(1 + m/r_0)^3} \tag{16}$$

The mass density, which is equal to the charge density, in the interior can be calculated from (11) and is given by

$$4\pi\rho = \frac{3m}{r_0{}^3}\left(1+\frac{m}{r_0}\right)^{-3}\left(1+\frac{mr^2}{r_0{}^3}\right)^{-1} \tag{17}$$

The mass density is maximum at the origin $r=0$ and decreases outwards with positive value everywhere. Eventually the solutions given by (14) and (16) are exactly the one obtained previously by Bonnor and Wickramsuriya (1975) as a special case of spherically symmetric charged dust. Our results are more general in the sense that the above solution is the only static conformally flat solution corresponding to charged dust irrespective of any a priori assumption on its symmetry.

The manifest form of the metric conformal to the Minskowskian metric may be obtained from equation (14) by the following transformations (Gurses and Gursey, 1975): With

$$\begin{aligned} \frac{r}{B^{1/2}} &= \frac{1}{A^{1/2}}\frac{\sin\rho}{1+\cos\rho} \\ B^{1/2}t &= \frac{1}{2A^{1/2}}\Upsilon \end{aligned} \tag{18}$$

the line element reduces to

$$ds^2 = \frac{1}{2A(1+\cos\rho)}\left[d\Upsilon^2 - d\rho^2 - \sin^2\rho\, d\Omega^2\right] \tag{19}$$

Again, with the transformation

$$\Upsilon = \arc\tan\left[\frac{A^{1/2}\eta}{1-(A/4)(\eta^2-\xi^2)}\right], \qquad \rho = \arc\tan\left[\frac{A^{1/2}\xi}{1+(A/4)(\eta^2-\xi^2)}\right]$$

the line element (19) finally takes the form

$$ds^2 = f(\xi,\eta)(d\eta^2 - d\xi^2 - \xi^2 d\Omega^2) \tag{20}$$

where the conformal factor $f(\xi,\eta)$ is given by

$$\begin{aligned} f(\xi,\eta) = \frac{1}{2}\left\{\left[1+\frac{A}{2}(\eta^2+\xi^2)+\frac{A^2}{16}(\eta^2+\xi^2)\right]^{1/2} + \left[1+\frac{A}{4}(\eta^2-\xi^2)\right]\right\}^{-1} \\ \times\left[1+\frac{A}{2}(\eta^2+\xi^2)+\frac{A^2}{16}(\eta^2-\xi^2)\right]^{-1/2} \end{aligned}$$

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